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Box ball system associated with antisymmetric tensor crystals

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Abstract

A new box ball system associated with an antisymmetric tensor crystal of the quantum affine algebra of type A is considered. This includes the so-called coloured box ball system with capacity 1 as the simplest case. An infinite number of conserved quantities is constructed and the scattering rule of two solitons is given explicitly.

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1. Introduction

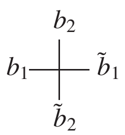
The box ball system (BBS) [TS, T] is an important example of ultra-discrete integrable systems. It can be obtained from a discrete soliton equation through limiting procedure [TTMS, TNS]. It is also known [HHIKTT, FOY, HKT, HKOTY] that the BBS has an equivalent formulation in terms of affine crystal theory [K, KMN]. Following [FOY] we briefly review the BBS in this formulation. For $l \in \mathbf{Z}_{\geqslant 1}$ set

$$B_l = \{[i_1 i_2 \cdots i_l] \mid i_k \in \mathbf{Z}, 1 \leqslant i_1 \leqslant \cdots \leqslant i_l \leqslant n\}.$$

Here n is a fixed integer greater than 1. B_l is identified with the crystal of the symmetric tensor representation of degree l of the quantum affine algebra $U'_q(\widehat{sl}_n)$. By definition, the combinatorial R for the tensor product crystal $B_{l_1} \otimes B_{l_2}$ is a bijection

$$R : B_{l_1} \otimes B_{l_2} \longrightarrow B_{l_2} \otimes B_{l_1}$$

which commutes with the crystal operators [KMN]. Use the symbol



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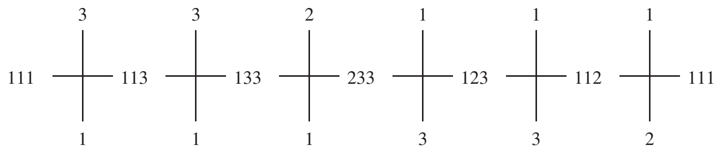
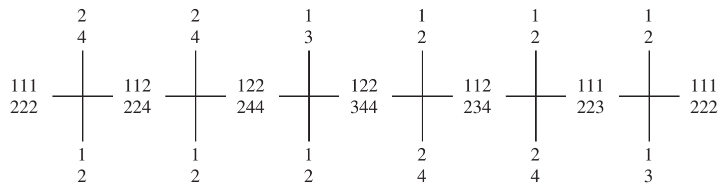
if $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$. Applying the combinatorial $R : B_3 \otimes B_1 \to B_1 \otimes B_3$ successively, we have the following diagram:



Here we omitted the symbol [ ]. For instance, 113 (resp. 2) means [113] (resp. [2]). Neglecting the horizontal line, we see the upper state is mapped to the lower one as the time is evolved. Each $[\cdot]$ corresponds to a box. We consider [1] as a vacant box, and [2], [3], ... as boxes with a ball. The above example shows that the array [3][3][2] behaves as a soliton just as in the classical soliton theory. One can also check that the longer soliton moves faster. Therefore, if there are two solitons of different lengths, we can expect a scattering of solitons. The following results are shown in [FOY] (see also [TNS, HHIKTT]).

(1) For an element $[i_1 i_2 \cdots i_l]$ of B_l such that $i_1 \geqslant 2$, one can associate a soliton state $[i_l] \cdots [i_2][i_1]$.
(2) The scattering of two solitons of lengths l_1 and l_2 ($l_1 > l_2$) is described by the combinatorial $R : B_{l_1} \otimes B_{l_2} \to B_{l_2} \otimes B_{l_1}$.
(3) The phase shift caused by the scattering is described by the energy function, an integer-valued function canonically defined on the tensor product crystal $B_{l_1} \otimes B_{l_2}$.

We want to extend these results to more general cases. It is known that the set of semi-standard tableaux $B^{k,l}$ of shape (l^k) with letters in $\{1, 2, \ldots, n\}$ also admits the affine crystal structure. Considering the combinatorial $R : B^{2,3} \otimes B^{2,1} \to B^{2,1} \otimes B^{2,3}$, we obtain the following:



As seen in this example, the situation for general k is similar to the previous case corresponding to $k = 1$. In this paper, we are to show the following results:

(1) One can associate a soliton state for an element of $B^{k-1,l}$(letters in $\{1, 2, \ldots, k\}$) $\times$ $B^{1,l}$(letters in $\{k+1, k+2, \ldots, n\}$).
(2) The scattering of two solitons of lengths l_1 and l_2 ($l_1 > l_2$) is described by the product of combinatorial R

$$R \times R : (B^{k-1,l_1} \otimes B^{k-1,l_2}) \times (B^{1,l_1} \otimes B^{1,l_2}) \longrightarrow (B^{k-1,l_2} \otimes B^{k-1,l_1}) \times (B^{1,l_2} \otimes B^{1,l_1}).$$

(3) The phase shift is described by the sum of the energy functions corresponding to the tensor product crystals $B^{k-1,l_1} \otimes B^{k-1,l_2}$ and $B^{1,l_1} \otimes B^{1,l_2}$.

2. Box ball system

In this section we review the minimum contents concerned with the $U_q'(\widehat{sl}_n)$-crystal $B^{k,l}$ and define the box ball system we shall investigate.

2.1. *The crystal $B^{k,l}$*

Fix an integer $n \in \mathbf{Z}_{\geqslant 2}$. For arbitrary integers k, l such that $1 \leqslant k \leqslant n-1, l \geqslant 1$, let $B^{k,l}$ be the set of semi-standard tableaux of shape (l^k) with letters in $\{1, 2, \ldots, n\}$. A tableau is described by using [ ] and ${}^t[\,]$ signifies its transpose. For instance,

$$\begin{bmatrix}1\\3\\5\end{bmatrix} \in B^{3,1}, \qquad \begin{bmatrix}1&1&2\\2&3&3\end{bmatrix} \in B^{2,3} \qquad \text{and} \qquad {}^t\begin{bmatrix}1&1&2\\2&3&3\end{bmatrix} = \begin{bmatrix}1&2\\1&3\\2&3\end{bmatrix} \in B^{3,2}.$$

It is known [KMN2, S] that $B^{k,l}$ admits the structure of $U_q'(\widehat{sl}_n)$-crystals, i.e., one has the action of operators e_i, f_i, called Kashiwara operators,

$$e_i, f_i : B^{k,l} \longrightarrow B^{k,l} \sqcup \{0\} \qquad \text{for} \quad i = 0, 1, \ldots, n-1.$$

We are to explain the actions of e_i, f_i on $B^{k,l}$ for $i \neq 0$ in detail. To do this we first give the actions on $(B^{k,1})^{\otimes L}$, Lth tensor power of $B^{k,1}$. The rule is given as follows:

(1) We identify $b = \otimes_{j=1}^{L} b_j \in (B^{k,1})^{\otimes L}$ with $b' \in (B^{1,1})^{\otimes kL}$ in such a way that

$$b' = \bigotimes_{j=1}^{L}\left(\bigotimes_{m=1}^{k}\left[x_m^j\right]\right) \qquad \text{if} \quad b_j = {}^t\left[x_1^j, x_2^j, \ldots, x_k^j\right].$$

Here and in what follows $\otimes_{j=1}^{L} b_j$ means $b_1 \otimes b_2 \otimes \cdots \otimes b_L$.

(2) Reading the letters x_m^j in b' from left to right, we construct a sequence of $\pm$ or 0 by associating + if $x_m^j = i$, $-$ if $x_m^j = i+1$, and 0 otherwise.

(3) Neglecting all 0, replace adjacent $+-$ pairs with 00 successively until we get the following sequence:

$$- - \cdots - - + + + \cdots + +$$

(4) The action of e_i (resp. f_i) on b amounts to replacing the rightmost $-$ (resp. leftmost +) of the above sequence with + (resp. $-$), i.e., changing the letter $i+1$ (resp. i) to i (resp. $i+1$) on the corresponding tensor component of b'.

Example 2.1.

$$f_2\left(\begin{bmatrix}1\\2\end{bmatrix} \otimes \begin{bmatrix}2\\3\end{bmatrix} \otimes \begin{bmatrix}4\\6\end{bmatrix}\right) = f_2(122346) = f_2(0++-00) = f_2(0+0000)$$
$$= 0-0000 = \begin{bmatrix}1\\3\end{bmatrix} \otimes \begin{bmatrix}2\\3\end{bmatrix} \otimes \begin{bmatrix}4\\6\end{bmatrix}.$$

To compute the action on $B^{k,l}$ we consider the map

$$\mathrm{sp} : B^{k,l} \longrightarrow (B^{k,1})^{\otimes l} \qquad b = [x^1, x^2, \ldots, x^l] \mapsto x^l \otimes \cdots \otimes x^2 \otimes x^1,$$

where x^j stands for the jth column of b. Then the action of e_i (resp. f_i) on $B^{k,l}$ is given by $\mathrm{sp}^{-1} \circ e_i \circ \mathrm{sp}$ (resp. $\mathrm{sp}^{-1} \circ f_i \circ \mathrm{sp}$).

Remark 2.2. For the action of e_0, f_0 we refer to [S], since it is unnecessary for our purpose.

This rule, called the signature rule, can be applicable to any tensor product of the form $B^{k_1,l_1} \otimes \cdots \otimes B^{k_d,l_d}$ by embedding it into $(B^{k_1,1})^{\otimes l_1} \otimes \cdots \otimes (B^{k_d,1})^{\otimes l_d}$ using $\mathrm{sp}^{\otimes d}$. Although it is inhomogeneous in the sense that k_j can vary, the signature rule holds as it is.

2.2. Combinatorial R and energy function

The combinatorial R for the tensor product crystal $B_1 \otimes B_2$ is, by definition, a bijection $R : B_1 \otimes B_2 \longrightarrow B_2 \otimes B_1$ which commutes with the operators e_i and f_i for all i. For our case concerning the $U'_q(\widehat{sl}_n)$-crystal $B^{k,l}$, such R is known to exist uniquely. To describe the combinatorial R explicitly, we explain Schensted's bumping algorithm. See, e.g., [F] for the details. For

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^l \\ x_2^1 & x_2^2 & \cdots & x_2^l \\ \vdots & \vdots & \ddots & \vdots \\ x_k^1 & x_k^2 & \cdots & x_k^l \end{bmatrix} \in B^{k,l},$$

we define a row word $\mathrm{row}(x)$ by

$$\mathrm{row}(x) = \underbrace{x_k^1 \ldots x_k^l}_{x_k} \cdots\cdots\cdots \underbrace{x_2^1 \ldots x_2^l}_{x_2} \underbrace{x_1^1 \ldots x_1^l}_{x_1}.$$

Now we recall the bumping algorithm (row-bumping, or row-insertion), for constructing a new tableau from a tableau by inserting an integer. For inserting an integer i in a tableau T, the rule of bumping '$T \leftarrow i$' is given as follows:

(1) If there are no integers larger than i in the first row, add a new empty box at the right end, and put i in it.
(2) Otherwise, among the integers larger than i, find the leftmost one, say j, and replace j with i. Then inset j into the second row in the same way.
(3) Repeat this procedure until the bumped number can be put in a new box at the right end of the row.

The following is the explicit algorithm of combinatorial R on $B^{k,l} \otimes B^{k',l'}$.

Theorem 2.3 ([S]). *$x \otimes y$ is mapped to $\tilde{x} \otimes \tilde{y}$ by the combinatorial R*

$$B^{k,l} \otimes B^{k',l'} \longrightarrow B^{k',l'} \otimes B^{k,l},$$

if and only if

$$y \leftarrow \mathrm{row}(x) = \tilde{y} \leftarrow \mathrm{row}(\tilde{x}).$$

Next, we define the energy function H on $B^{k,l} \otimes B^{k',l'}$ as follows.

Definition 2.4. *Let $x \in B^{k,l}$, $y \in B^{k',l'}$. Let $d(x, y)$ be the number of nodes in the shape of $y \leftarrow \mathrm{row}(x)$ that are strictly east of the $\max(l, l')$th column. Then the energy function $H : B^{k,l} \otimes B^{k',l'} \to \mathbf{Z}$ is given by*

$$H(x \otimes y) = d(x, y) - \min(k, k')\min(l, l').$$

Remark 2.5. Intrinsically, the energy function is uniquely defined by specifying the difference $H(e_i(x \otimes y)) - H(x \otimes y)$ up to an additive constant. See section 4.1 of [KMN]. The above rule for the energy function is established in [S, SW]. The normalization of H in definition 2.4 is so fixed that the maximum is 0.

Here we give an example for combinatorial R on $B^{3,3} \otimes B^{2,1}$ and energy function associated with this R.

Example 2.6.

$$\begin{bmatrix}1&2&4\\2&3&5\\4&4&6\end{bmatrix}\otimes\begin{bmatrix}2\\5\end{bmatrix}\simeq\begin{bmatrix}2\\4\end{bmatrix}\otimes\begin{bmatrix}1&2&3\\2&4&5\\4&5&6\end{bmatrix}.$$

We will show the procedure for calculating the right-hand side when left-hand side is given. We set

$$x=\begin{bmatrix}1&2&4\\2&3&5\\4&4&6\end{bmatrix},\qquad y=\begin{bmatrix}2\\5\end{bmatrix}.$$

By the rule of bumping we have the following tableau.

$$y\leftarrow \mathrm{row}(x)=\begin{matrix}2\\5\end{matrix}\quad\leftarrow 446235124=\begin{matrix}1&2&2&4\\2&3&5&\\4&4&6&\\5&&&\end{matrix}.$$

Note that the rule of bumping is a reversible procedure, and we can calculate $\tilde{x}\in B^{2,1}$ and $\tilde{y}\in B^{3,3}$ such that $x\otimes y\simeq\tilde{x}\otimes\tilde{y}$.

$$\begin{matrix}1&2&2&4\\2&3&5&\\4&4&6&\\5&&&\end{matrix}=\begin{matrix}1&2&2&4\\2&4&5&\\4&4&6&\end{matrix}\leftarrow 4=\begin{matrix}1&2&2&4\\2&4&5&\\4&4&6&\end{matrix}\leftarrow 3=\begin{matrix}1&2&3&4\\2&4&5&\\4&4&6&\end{matrix}\leftarrow 2$$

$$=\begin{matrix}1&2&3\\2&4&5\\4&4&6\end{matrix}\leftarrow 42=\tilde{y}\leftarrow\mathrm{row}(\tilde{x}).$$

According to the definition of energy function, we have

$$H(x\otimes y)=1-\min(3,2)\min(3,1)=-1.$$

2.3. Box ball system

In this subsection we fix k $(1\leqslant k\leqslant n-1)$ and set $B_l=B^{k,l}$. We employ the following notation:

$$1_k=\begin{bmatrix}1\\2\\\vdots\\k\end{bmatrix}\in B_1,\qquad c_l=1_k^l=\begin{bmatrix}1&1&\cdots&1\\2&2&\cdots&2\\\vdots&\vdots&\ddots&\vdots\\k&k&\cdots&k\end{bmatrix}\in B_l.$$

Consider the crystal $B_1^{\otimes L}$ with L sufficiently large. We call an element p of $B_1^{\otimes L}$ a state, if

$$p=\left(\bigotimes_{j=1}^{d}b_j\right)\otimes 1_k^{\otimes(L-d)}$$

and L is sufficiently large compared with d. Note that we allow 1_k to be in $\otimes_{j=1}^{d}b_j$.

Lemma 2.7. *By iterating the combinatorial* $R: B_l\otimes B_1\to B_1\otimes B_l$ L *times, we have the map*

$$B_l\otimes B_1^{\otimes L}\longrightarrow B_1^{\otimes L}\otimes B_l.$$

Suppose $c_l\otimes p$ *is mapped to* $\tilde{p}\otimes c$ *by this map. Then, if* $L\gg 1$, *we have* $c=c_l$.

By this lemma one can define the time evolution operator T_l by

$$T_l : B_1^{\otimes L} \ni p \mapsto \tilde{p} \in B_1^{\otimes L}.$$

Then we have

Lemma 2.8. $[T_l, e_i] = [T_l, f_i] = 0$ *for* $i \in I\backslash\{0, k\}$.

Proof. By the definition of T_l, we have

$$c_l \otimes e_i p \mapsto T_l(e_i p) \otimes c_l, \qquad e_i(c_l \otimes p) \mapsto e_i(T_l(p) \otimes c_l),$$

under the map $B_l \otimes B_1^{\otimes L} \to B_1^{\otimes L} \otimes B_l$. On the other hand, for $i \in I\backslash\{0, k\}$ using the signature rule in section 2.1 we have

$$e_i(c_l \otimes p) = c_l \otimes e_i p, \qquad e_i(T_l(p) \otimes c_l) = e_i T_l(p) \otimes c_l.$$

Thus we have $T_l(e_i p) \otimes c_l = e_i T_l(p) \otimes c_l$. The case for f_i is similar. □

This property will be used essentially to prove our main theorem.

We also define a map $E_l : B_1^{\otimes L} \to \mathbf{Z}_{\geqslant 0}$ by

$$E_l(p) = -\sum_{j=1}^{L} H(b^{(j-1)} \otimes b_j), \tag{2.1}$$

where $b^{(j)}$ $(0 \leqslant j < L)$ is defined by

$$B_l \otimes B_1^{\otimes j} \ni c_l \otimes \left(\bigotimes_{i=1}^{j} b_i\right) \mapsto \left(\bigotimes_{i=1}^{j} \tilde{b}_i\right) \otimes b^{(j)} \in B_1^{\otimes j} \otimes B_l$$

and H is the energy function on $B_l \otimes B_1$. Note that for a state p, $E_l(p)$ does not depend on L, because of lemma 2.7 and the normalization $H\left(1_k^l \otimes 1_k\right) = 0$.

It is known that the following equation, called the Yang–Baxter equation, holds on $B_l \otimes B_{l'} \otimes B_{l''}$.

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$$

From this fact one can show

Theorem 2.9 ([FOY])**.**

(1) The commutation relation among the time evolutions: $T_l T_{l'}(p) = T_{l'} T_l(p)$.

(2) The energy conservation: $E_l(T_{l'}(p)) = E_l(p)$.

To be precise, we need the Yang–Baxter equation for the affinization of $B_l, B_{l'}, B_{l''}$ to prove (2). See section 2.4 of [FOY].

In what follows in this section, we present other conserved quantities $P_{\leqslant k}, P_{>k}$ under time evolutions constructed by Schensted's bumping algorithm. By the isomorphism $B_l \otimes B_1^{\otimes L} \simeq B_1^{\otimes L} \otimes B_l$, we have

$$c_l \otimes \left(\bigotimes_{j=1}^{L} b_j\right) \mapsto \left(\bigotimes_{j=1}^{L} \tilde{b}_j\right) \otimes c_l.$$

In view of theorem 2.3 one notes that the tableau constructed by

$$((\cdots(\mathrm{row}(b_L) \leftarrow \mathrm{row}(b_{L-1})) \leftarrow \cdots) \leftarrow \mathrm{row}(b_1)) \leftarrow \mathrm{row}(c_l)$$

and the one by

$$((\cdots(\mathrm{row}(c_l) \leftarrow \mathrm{row}(\tilde{b}_L))\cdots) \leftarrow \mathrm{row}(\tilde{b}_2)) \leftarrow \mathrm{row}(\tilde{b}_1)$$

are the same. It means that the corresponding words are Knuth-equivalent [F].

$$\mathrm{row}(b_L)\,\mathrm{row}(b_{L-1})\cdots\mathrm{row}(b_1)\,\mathrm{row}(c_l)\sim\mathrm{row}(c_l)\,\mathrm{row}(\tilde{b}_L)\cdots\mathrm{row}(\tilde{b}_2)\,\mathrm{row}(\tilde{b}_1). \tag{2.2}$$

Remove the numbers $k+1, k+2, \ldots, n$ from both sequences; then they are still Knuth-equivalent. (See, e.g., lemma 1 of [Fu].) Thus the tableaux constructed by both words are the same. It gives a conserved quantity, denoted by $P_{\leqslant k}$, under time evolutions. Note that taking $\mathrm{row}(c_l)$ in the words corresponds to adding c_l from the left at the construction of tableaux. Similarly, by removing the numbers $1, 2, \ldots, k$ from (2.2), we obtain another conserved quantity $P_{>k}$. Note that the P-tableau in theorem 3.1 of [Fu] corresponds to $P_{>k}$.

3. Solitons

In the previous section we introduced a box ball system on the crystal $(B^{k,1})^{\otimes L}$. In this section we define soliton states and investigate the scattering of two solitons.

3.1. One-soliton state

Consider the following state:

$$p = 1_k^{\otimes c}\otimes\left(\bigotimes_{j=1}^{d} b_j\right)\otimes 1_k^{\otimes(L-c-d)}\in(B^{k,1})^{\otimes L}, \qquad b_j = {}^t\left[x_1^j, x_2^j, \ldots, x_k^j\right]\in B^{k,1}.$$

Definition 3.1. *We call such a* p *a one-soliton state if the integers* $x_i^j \in \{1, 2, \ldots, n\}$ $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant d)$ *satisfy*

$$\begin{aligned}
&x_i^j \geqslant x_i^{j+1} && (1\leqslant i\leqslant k, 1\leqslant j\leqslant d-1),\\
&x_i^j < x_{i+1}^j && (1\leqslant i\leqslant k-2, 1\leqslant j\leqslant d),\\
&x_{k-1}^j \leqslant k < x_k^j && (1\leqslant j\leqslant d).
\end{aligned}$$

We call d the length of the soliton.

Remark 3.2. Such a state is characterized by the condition $E_1(p) = 1$.

Example 3.3. In the case of $k = 3$, the following gives a one-soliton state of length 3.

$$\cdots\otimes\begin{bmatrix}1\\2\\3\end{bmatrix}\otimes\begin{bmatrix}2\\3\\5\end{bmatrix}\otimes\begin{bmatrix}2\\3\\4\end{bmatrix}\otimes\begin{bmatrix}1\\2\\4\end{bmatrix}\otimes\begin{bmatrix}1\\2\\3\end{bmatrix}\otimes\cdots$$

To simplify the notation, we use

$$\begin{aligned}
&[\mathbf{0}] = {}^t[1,\ldots,k-2,k-1,k],\\
&[\mathbf{1}] = {}^t[1,\ldots,k-2,k-1,k+1],\\
&[\mathbf{2}_\pm] = {}^t\left[1,\ldots,k-2,k-\tfrac{1}{2}\pm\tfrac{1}{2},k+\tfrac{3}{2}\mp\tfrac{1}{2}\right],\\
&[\mathbf{3}] = {}^t[1,\ldots,k-2,k,k+2],\\
&[\mathbf{4}] = {}^t[1,\ldots,k-2,k+1,k+2].
\end{aligned}$$

The following lemma shows how a one-soliton state behaves as time evolves.

Lemma 3.4. *Let* $p = 1_k^{\otimes c} \otimes S_d \otimes 1_k^{\otimes(L-c-d)}$ *be a one-soliton state of length d. Then,*

$$T_l^t(p) = 1_k^{\otimes(c+\min(d,l)t)} \otimes S_d \otimes 1_k^{\otimes(L-c-d-\min(d,l)t)}.$$

Proof. By lemma 2.8, it is sufficient to check for the elements killed by e_i for all $i \in I\backslash\{0, k\}$, i.e.

$$p = [\mathbf{0}]^{\otimes c} \otimes [\mathbf{1}]^{\otimes d} \otimes [\mathbf{0}]^{\otimes(L-c-d)}.$$

Our $R : B_l \otimes B_1 \rightarrow B_1 \otimes B_l$ satisfies

$$R(\xi_i \otimes [\mathbf{1}]) = [\mathbf{0}] \otimes \xi_{i+1} \quad (0 \leqslant i < l), \qquad R(\xi_l \otimes [\mathbf{1}]) = [\mathbf{1}] \otimes \xi_l,$$
$$R(\xi_i \otimes [\mathbf{0}]) = [\mathbf{1}] \otimes \xi_{i-1} \quad (0 < i \leqslant l), \qquad R(\xi_0 \otimes [\mathbf{0}]) = [\mathbf{0}] \otimes \xi_0,$$

where $\xi_i = [\mathbf{0}^{l-i}\,\mathbf{1}^i]$ $(0 \leqslant i \leqslant l)$. The statement is clear from these formulae. □

We call the value c the phase of the soliton S_d. Note that the phase is related to the position of the soliton *before* applying time evolutions. Hence it does not depend on time for a one-soliton state.

3.2. The scattering rule of solitons

Imitating the definition of one-soliton state, we can consider an m soliton state

$$p_{ms} = \bigotimes_{j=1}^{m} \left(1_k^{\otimes(c_j-c_{j-1}-d_{j-1})} \otimes S_{d_j}\right) \otimes 1_k^{\otimes(L-c_m-d_m)}, \tag{3.1}$$

where $c_0 = d_0 = 0$. S_{d_j} signifies a soliton of length d_j satisfying the conditions in definition 3.1, and we assume $c_j - c_{j-1} - d_{j-1}$ $(1 \leqslant j \leqslant m)$ are relatively large compared with d_j $(j = 1, \ldots, m)$.

For a single soliton we introduce new notation. One is related to the internal degree of freedom. For $S_d = \otimes_{j=1}^d b_j$ we set $u = [b_d, \ldots, b_2, b_1]$. Note that from definition 3.1 u can be regarded as a tableau in $B^{k,l}$. Next, we split u between the $(k-1)$th and kth row as

$$u = (u_{<k}, u_k) \in B^{k-1,l} \times B^{1,l}. \tag{3.2}$$

Namely u_k is the kth row of u and $u_{<k}$ is the tableau of $k-1$ rows obtained by removing the kth row of u. Note that $u_{<k}$ (resp. u_k) is a tableau with letters in $\{1, 2, \ldots, k\}$ (resp. $\{k+1, k+2, \ldots, n\}$). Thus $B^{k-1,l}$ (resp. $B^{1,l}$) in (3.2) should be considered as a $U_q(sl_k)(= \langle e_i, f_i, t_i\ (i = 1, 2, \ldots, k-1)\rangle)$-crystal (resp. $U_q(sl_{n-k})(= \langle e_i, f_i, t_i\ (i = k+1, k+2, \ldots, n-1)\rangle)$-crystal). The other is related to the phase explained at the end of the previous subsection. If the soliton with the internal degree of freedom u has a phase c, we represent it as $\zeta^c u$ with ζ indeterminate.

Next we consider a two-soliton state. Let $\zeta^{c_1}u$ and $\zeta^{c_2}v$ be solitons with the notation above. We assume the length d_1 of the soliton $\zeta^{c_1}u$ is bigger than the length d_2 of $\zeta^{c_2}v$. We also assume that the former is situated far left to the latter at time $t = 0$. We represent such a situation as $\zeta^{c_1}u \otimes \zeta^{c_2}v$. Now apply T_l $(l > d_2)$ many times. Then one observes that the longer soliton passes the shorter one and the phases change by a constant. We call this phenomenon scattering of two solitons and represent it as

$$\zeta^{c_1}u \otimes \zeta^{c_2}v \longrightarrow \zeta^{c_2-\delta_2}\tilde{v} \otimes \zeta^{c_1+\delta_1}\tilde{u}.$$

Then the following gives the scattering rule for such two solitons.

Theorem 3.5. *The scattering rule of two-soliton state* p_{2s} $(d_1 > d_2)$ *under the time evolution* T_l $(l > d_2)$ *is described as follows. The change of the internal degree of freedom is given by the product of combinatorial R's* $R \times R$*:*

$$(B^{k-1,d_1} \otimes B^{k-1,d_2}) \times (B^{1,d_1} \otimes B^{1,d_2}) \longrightarrow (B^{k-1,d_2} \otimes B^{k-1,d_1}) \times (B^{1,d_2} \otimes B^{1,d_1})$$
$$(u_{<k} \otimes v_{<k}, u_k \otimes v_k) \mapsto (\tilde{v}_{<k} \otimes \tilde{u}_{<k}, \tilde{v}_k \otimes \tilde{u}_k).$$

The phase shifts δ_1, δ_2 *are given by the sum of the energy functions* H *associated with each combinatorial* R*:*

$$\delta_1 = \delta_2 = 2d_2 + H(u_k \otimes v_k) + H(u_{<k} \otimes v_{<k}).$$

The proof will be given in the next subsection.

Remark 3.6. For a two-soliton state p_{2s} we regard $c_2 - c_1 - d_1$ as the distance from S_{d_1} to S_{d_2}. The above scattering rule of two solitons is valid if $c_2 - c_1 - d_1 \geqslant d_2$ holds at the initial stage.

Remark 3.7. The image $(R \times R)(u_{<k} \otimes v_{<k}, u_k \otimes v_k)$ is different from $R(u \otimes v)$ by $R : B^{k,d_1} \otimes B^{k,d_2} \to B^{k,d_2} \otimes B^{k,d_1}$. For instance, we have

$$(R \times R)\left(\begin{bmatrix}1&1&1&1&2\\2&2&3&3&3\end{bmatrix} \otimes \begin{bmatrix}1&1&2\\2&3&3\end{bmatrix}, [4\ \ 4\ \ 4\ \ 5\ \ 5] \otimes [5\ \ 6\ \ 7]\right)$$
$$= \left(\begin{bmatrix}1&1&2\\2&3&3\end{bmatrix} \otimes \begin{bmatrix}1&1&1&1&2\\2&2&3&3&3\end{bmatrix}, [4\ \ 5\ \ 5] \otimes [4\ \ 4\ \ 5\ \ 6\ \ 7]\right),$$

whereas

$$R\left(\begin{bmatrix}1&1&1&1&2\\2&2&3&3&3\\4&4&4&5&5\end{bmatrix} \otimes \begin{bmatrix}1&1&2\\2&3&3\\5&6&7\end{bmatrix}\right) = \begin{bmatrix}1&1&2\\3&3&3\\4&5&5\end{bmatrix} \otimes \begin{bmatrix}1&1&1&1&2\\2&2&2&4&4\\3&3&5&6&7\end{bmatrix}.$$

Recall the scattering rule of two solitons is written as

$$\zeta^{c_1}\begin{bmatrix}u_{<k}\\u_k\end{bmatrix} \otimes \zeta^{c_2}\begin{bmatrix}v_{<k}\\v_k\end{bmatrix} \to \zeta^{c_2-\delta}\begin{bmatrix}\tilde{v}_{<k}\\\tilde{v}_k\end{bmatrix} \otimes \zeta^{c_1+\delta}\begin{bmatrix}\tilde{u}_{<k}\\\tilde{u}_k\end{bmatrix}$$

$$\delta = 2d_2 + H(u_k \otimes v_k) + H(u_{<k} \otimes v_{<k}).$$

Denote this map by $\tilde{R}$. Then it satisfies the Yang–Baxter equation

$$(\tilde{R} \otimes 1)(1 \otimes \tilde{R})(\tilde{R} \otimes 1) = (1 \otimes \tilde{R})(\tilde{R} \otimes 1)(1 \otimes \tilde{R}). \tag{3.3}$$

It means that the scattering rule of three solitons is independent of the order of two body ones. It is so even when three solitons collide almost at the same time. We check this property with the following example. Here we employ the notation

$$(x_1, x_2, \ldots, x_n) = \bigotimes_{j=1}^{n} x_j \otimes 1_k \otimes 1_k \otimes \cdots.$$

Example 3.8.

$$p_{3s} = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 3 \\ 5 & 4 & 4 & 3 & 3 & 3 & 6 & 4 & 3 & 3 & 3 & 5 \end{pmatrix}$$

$$T_3^1(p_{3s}) = 1_3^{\otimes 3} \otimes \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 2 & 3 \\ 5 & 4 & 4 & 3 & 3 & 6 & 4 & 3 & 3 & 5 \end{pmatrix}$$

$$T_3^2(p_{3s}) = 1_3^{\otimes 6} \otimes \begin{pmatrix} 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 3 & 2 & 2 & 3 \\ 5 & 4 & 4 & 3 & 6 & 4 & 3 & 5 \end{pmatrix}$$

$$T_3^3(p_{3s}) = 1_3^{\otimes 9} \otimes \begin{pmatrix} 2 & 2 & 1 & 2 & 2 & 1 \\ 3 & 3 & 2 & 3 & 3 & 4 \\ 5 & 4 & 3 & 6 & 4 & 5 \end{pmatrix}$$

$$T_3^4(p_{3s}) = 1_3^{\otimes 11} \otimes \begin{pmatrix} 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 4 & 3 & 3 \\ 5 & 4 & 3 & 3 & 6 & 5 & 4 \end{pmatrix}$$

$$T_3^5(p_{3s}) = 1_3^{\otimes 13} \otimes \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 2 & 3 & 3 & 3 \\ 5 & 4 & 3 & 4 & 3 & 6 & 5 & 4 \end{pmatrix}$$

$$T_3^6(p_{3s}) = 1_3^{\otimes 15} \otimes \begin{pmatrix} 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\ 3 & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 \\ 5 & 3 & 4 & 4 & 3 & 3 & 6 & 5 & 4 \end{pmatrix}.$$

With our notation this three-body scattering is expressed as

$$\zeta^0 \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 5 \end{bmatrix} \otimes \zeta^6 \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} \otimes \zeta^{11} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
$$\longrightarrow \zeta^9 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \otimes \zeta^5 \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 4 \end{bmatrix} \otimes \zeta^3 \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Let us check that the RHS is obtained by applying either $(\tilde{R} \otimes 1)(1 \otimes \tilde{R})(\tilde{R} \otimes 1)$ or $(1 \otimes \tilde{R})(\tilde{R} \otimes 1)(1 \otimes \tilde{R})$ on the LHS.

$$\text{LHS} \xrightarrow{\tilde{R}\otimes 1} \zeta^{6-3} \begin{bmatrix} 2 & 2 \\ 3 & 3 \\ 4 & 5 \end{bmatrix} \otimes \zeta^{0+3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 4 & 4 & 6 \end{bmatrix} \otimes \zeta^{11} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
$$\xrightarrow{1\otimes \tilde{R}} \zeta^3 \begin{bmatrix} 2 & 2 \\ 3 & 3 \\ 4 & 5 \end{bmatrix} \otimes \zeta^{11-0} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \otimes \zeta^{3+0} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$\xrightarrow{\tilde{R}\otimes 1} \zeta^{11-2} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \otimes \zeta^{3+2} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 4 \end{bmatrix} \otimes \zeta^3 \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{aligned}
\text{LHS} &\xrightarrow{1\otimes\tilde{R}} \zeta^{0}\begin{bmatrix}2&2&2\\3&3&3\\4&4&5\end{bmatrix}\otimes\zeta^{11-0}\begin{bmatrix}1\\2\\4\end{bmatrix}\otimes\zeta^{6+0}\begin{bmatrix}1&2\\3&3\\5&6\end{bmatrix}\\
&\xrightarrow{\tilde{R}\otimes 1} \zeta^{11-2}\begin{bmatrix}2\\3\\5\end{bmatrix}\otimes\zeta^{0+2}\begin{bmatrix}1&2&2\\2&3&3\\4&4&4\end{bmatrix}\otimes\zeta^{6}\begin{bmatrix}1&2\\3&3\\5&6\end{bmatrix}\\
&\xrightarrow{1\otimes\tilde{R}} \zeta^{9}\begin{bmatrix}2\\3\\5\end{bmatrix}\otimes\zeta^{6-1}\begin{bmatrix}1&2\\2&3\\4&4\end{bmatrix}\otimes\zeta^{2+1}\begin{bmatrix}1&2&2\\3&3&3\\4&5&6\end{bmatrix}.
\end{aligned}$$

3.3. Proof of the main theorem

By lemma 2.8 it is sufficient to show the theorem for the highest weight element p as a $U_q(sl_k)\times U_q(sl_{n-k})$-crystal, i.e., $e_i p = 0$ for all $i \in I\backslash\{0,k\}$. The following lemma is clear from the signature rule.

Lemma 3.9. *The highest weight elements among two-soliton states are given by*
$\tilde{p}_{2s} = [\mathbf{0}]^{\otimes c_1}\otimes[\mathbf{1}]^{\otimes d_1}\otimes[\mathbf{0}]^{\otimes(c_2-c_1-d_1)}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\otimes[\mathbf{1}]^{\otimes\beta}\otimes[\mathbf{0}]^{\otimes h_2}$,
where $h_2 = L - c_2 - d_2$ *and* $0 \leqslant \alpha+\beta \leqslant d_2$.

By the commutation relation of time evolutions we get

$$T_r^t = T_{d_2+1}^{-s}T_r^tT_{d_2+1}^{s} \qquad (t \gg s \gg 1, r > d_2).$$

Thus it is sufficient to show the scattering rule by the time evolution T_{d_2+1}. We give the time evolution process for the case $\alpha > \beta$ and $\alpha \leqslant \beta$ separately. Set $\delta = d_2 - \alpha + \beta$.

$$\begin{aligned}
T_{d_2+1}^t(\tilde{p}_{2s}) \quad &(\alpha > \beta)\\
&= [\mathbf{0}]^{\otimes(c_1+(d_2+1)t)}\otimes[\mathbf{1}]^{\otimes d_1}\otimes[\mathbf{0}]^{\otimes(c_2-c_1-d_1-t)}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\\
&\quad\otimes[\mathbf{1}]^{\otimes\beta}\otimes[\mathbf{0}]^{\otimes(h_2-d_2t)} \qquad (0\leqslant t\leqslant c_2-c_1-d_1)\\
&= [\mathbf{0}]^{\otimes(c_1+(d_2+1)t)}\otimes[\mathbf{1}]^{\otimes(d_1-s)}\otimes[\mathbf{4}]^{\otimes s}\otimes[\mathbf{3}]^{\otimes(\alpha-s)}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\\
&\quad\otimes[\mathbf{1}]^{\otimes\beta}\otimes[\mathbf{0}]^{\otimes(h_2-d_2t)} \qquad (0\leqslant s\leqslant\alpha-\beta, s=t-(c_2-c_1-d_1))\\
&= [\mathbf{0}]^{\otimes(c_1+(d_2+1)t)}\otimes[\mathbf{1}]^{\otimes(\delta-s)}\otimes[\mathbf{4}]^{\otimes(\alpha-\beta)}\otimes[\mathbf{3}]^{\otimes\beta}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}[\mathbf{1}]^{\otimes(\beta+s)}\\
&\quad\otimes[\mathbf{0}]^{\otimes(h_1-\delta-(d_2+1)t)} \qquad (0\leqslant s\leqslant d_1-d_2, s=t-(c_2-c_1+d_2-d_1-\delta))\\
&= [\mathbf{0}]^{\otimes(c_2-\delta+d_2t)}\otimes[\mathbf{1}]^{\otimes(\delta+s)}\otimes[\mathbf{4}]^{\otimes(\alpha-\beta-s)}\otimes[\mathbf{3}]^{\otimes(\beta+s)}\\
&\quad\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\otimes[\mathbf{1}]^{\otimes(\beta+d_1-d_2)}\otimes[\mathbf{0}]^{\otimes(h_1-\delta-(d_2+1)t)}\\
&\quad(0\leqslant s\leqslant\alpha-\beta, s=t-(c_2-c_1-\delta))\\
&= [\mathbf{0}]^{\otimes(c_2-\delta+d_2t)}\otimes[\mathbf{1}]^{\otimes d_2}\otimes[\mathbf{0}]^{\otimes s}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\otimes[\mathbf{1}]^{\otimes(d_1-d_2+\beta)}\\
&\quad\otimes[\mathbf{0}]^{\otimes(h_1-\delta-(d_2+1)t)} \qquad (0\leqslant s, s=t-(c_2-c_1+d_2-2\delta))\\
T_{d_2+1}^t(\tilde{p}_{2s}) \quad &(\alpha \leqslant \beta)\\
&= [\mathbf{0}]^{\otimes(c_1+(d_1+1)t)}\otimes[\mathbf{1}]^{\otimes d_1}\otimes[\mathbf{0}]^{\otimes(c_2-c_1-d_1-t)}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\\
&\quad\otimes[\mathbf{1}]^{\otimes\beta}\otimes[\mathbf{0}]^{\otimes(h_2-d_2t)} \qquad (0\leqslant t\leqslant c_2-c_1+d_2-d_1-\delta)\\
&= [\mathbf{0}]^{\otimes(c_1+(d_2+1)t)}\otimes[\mathbf{1}]^{\otimes(d_1-s)}\otimes[\mathbf{0}]^{\otimes(\beta-\alpha)}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\otimes[\mathbf{1}]^{\otimes(\beta+s)}\\
&\quad\otimes[\mathbf{0}]^{\otimes(h_1-\delta-(d_2+1)t)} \qquad (0\leqslant s\leqslant d_1-d_2, s=t-(c_2-c_1-\delta))\\
&= [\mathbf{0}]^{\otimes(c_2-\delta+d_2t)}\otimes[\mathbf{1}]^{\otimes d_2}\otimes[\mathbf{0}]^{\otimes(\beta-\alpha+s)}\otimes[\mathbf{3}]^{\otimes\alpha}\otimes[\mathbf{2}_\pm]^{\otimes(d_2-\alpha-\beta)}\\
&\quad\otimes[\mathbf{1}]^{\otimes(\beta+d_1-d_2)}\otimes[\mathbf{0}]^{\otimes(h_1-\delta-(d_2+1)t)} \qquad (0\leqslant s, s=t-(c_2-c_1-\delta)).
\end{aligned}$$

From these we have

$$\zeta^{c_1}[\mathbf{1}^{d_1}] \otimes \zeta^{c_2}\big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big] \to \zeta^{c_2-\delta}[\mathbf{1}^{d_2}] \otimes \zeta^{c_1+\delta}\big[\mathbf{1}^{d_1-d_2+\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big].$$

Recall the notation introduced in (3.2). From theorem 2.3 and definition 2.4, we get the following.

Lemma 3.10.

$$R\big([\mathbf{1}^{d_1}]_k \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_k\big) = [\mathbf{1}^{d_2}]_k \otimes \big[\mathbf{1}^{d_1-d_2+\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_k,$$

$$H\big([\mathbf{1}^{d_1}]_k \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_k\big) = \left(-\frac{d_2-\beta}{2} - \frac{\alpha}{2}\right) \mp \left(-\frac{d_2-\beta}{2} + \frac{\alpha}{2}\right),$$

$$R\big([\mathbf{1}^{d_1}]_{<k} \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_{<k}\big) = [\mathbf{1}^{d_2}]_{<k} \otimes \big[\mathbf{1}^{d_1-d_2+\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_{<k},$$

$$H\big([\mathbf{1}^{d_1}]_{<k} \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_{<k}\big) = \left(-\frac{d_2-\beta}{2} - \frac{\alpha}{2}\right) \pm \left(-\frac{d_2-\beta}{2} + \frac{\alpha}{2}\right).$$

From this lemma we get the phase shift as follows:

$$2d_2 + H\big([\mathbf{1}^{d_1}]_k \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_k\big) + H\big([\mathbf{1}^{d_1}]_{<k} \otimes \big[\mathbf{1}^{\beta}\mathbf{2}_{\pm}^{d_2-\alpha-\beta}\mathbf{3}^{\alpha}\big]_{<k}\big) = d_2 - \alpha + \beta = \delta,$$

and see that the change of the internal degree of freedom is given by the combinatorial R.

4. Discussion

In this paper we defined the one-soliton state $p \in B_1^{\otimes L}$ so that it satisfies $E_1(p) = 1$. See definition 3.1 and remark 3.2. But in the $E_1(p) > 1$ case also, there exist states in which a soliton-like particle moves without changing the internal degree of freedom. As such examples we have

$$(a)\begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} \qquad (b)\begin{bmatrix} 2 & 3 & 2 \\ 4 & 5 & 5 \end{bmatrix} \qquad (c)\begin{bmatrix} 2 & 1 & 2 & 3 & 2 \\ 4 & 3 & 4 & 5 & 5 \end{bmatrix}.$$

How do we regard these states? Aren't they one-soliton states? Our answer is no. For a state p define the numbers $N_d(p)$ $(d = 1, 2, \ldots)$ by solving

$$E_l(p) = \sum_{d \geqslant 1} \min(d, l) N_d(p) \qquad (l = 1, 2, \ldots),$$

where $E_l(p)$ was defined in (2.1). Then N_d gives the number of solitons of length d for an m soliton state

$$\ldots [d_1] \ldots [d_2] \ldots \cdots \ldots [d_m] \ldots.$$

if any adjacent solitons are separated enough. For the above examples the numbers N_d read

$$(a)\, N_1 = 2 \quad (b)\, N_2 = 2 \quad (c)\, N_2 = 3$$

and the others are 0. Thus one may consider them as composite states of several solitons. However, it is in general difficult to identify the internal degree of freedom of each soliton, and we leave it as a future problem.

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